Comment on "Time-averaged properties of unstable periodic orbits and chaotic orbits in ordinary differential equation systems"

Michael A. Zaks
Institut für Physik, Humboldt-Universität zu Berlin, D-12489 Berlin, Germany

Denis S. Goldobin
Department of Theoretical Physics, Perm State University, 614990 Perm, Russia

(Received 15 July 2009; published 26 January 2010)

A recent paper claims that mean characteristics of chaotic orbits differ from the corresponding values averaged over the set of unstable periodic orbits, embedded in the chaotic attractor. We demonstrate that the alleged discrepancy is an artifact of the improper averaging. Since the natural measure is nonuniformly distributed over the attractor, different periodic orbits make different contributions into the time averages. As soon as the corresponding weights are accounted for, the discrepancy disappears.

DOI: 10.1103/PhysRevE.81.018201 PACS number(s): 05.45.−a

Recent Rapid Communication [1] compares properties of unstable periodic orbits (UPOs) embedded into a chaotic attractor to those of chaotic trajectories on this attractor. Analysis is based on numerical data and culminates in the statement "time-averaged properties along a set of UPOs and a set of chaotic orbits with finite lengths are totally different from each other." It is further conjectured that "the time averages of the dynamical quantities along UPOs with the same period of the Poincaré map have a limiting distribution with nonzero variance." In this Comment, we show that under the proper averaging procedure, there seems to be neither the "total" difference between the averages nor an argument for the above conjecture.

Due to ergodicity, the value of time average for an observable A converges to $\bar{A}=\int A(\boldsymbol{x})\mu(\boldsymbol{x})d\boldsymbol{x}$, where μ is the natural measure, and integration is performed over the whole attractor. Approximation of $\bar{A}$ by summation over the set of UPOs—apparently the method employed in [1]—provides correct results only in exceptional cases when μ is uniformly distributed over the attractor: for linear mappings such as the Bernoulli map, symmetric tent map, etc. In general, however, the density of the natural measure varies along the attractor. As a consequence, a chaotic orbit does not walk uniformly over the attracting set: it visits certain regions relatively often or stays there relatively long. Contribution of such regions into the time averages is larger than of those visited seldom. As shown in [2] (see also Chapter 9.5 of the textbook [3]), nonuniform distribution of the natural measure can be recovered from the properties of UPOs embedded in the attractor. In particular, for invertible two-dimensional maps (and hence for three-dimensional flows which induce such maps), the weight with which an UPO contributes to the time average is inversely proportional to the largest eigenvalue of the corresponding fixed point. Below we demonstrate that taken the weights into account, the discrepancy between the mean values from the chaotic time series and the mean values from the UPOs disappears.

We restrict ourselves to the first example considered in [1]: attractor in the celebrated Lorenz equations

$$\frac{dx}{dt}=\sigma(y-x),\quad \frac{dy}{dt}=rx-y-xz,\quad \frac{dz}{dt}=xy-bz, \qquad (1)$$

with $\sigma=10$, $r=28$, $b=8/3$ [4] and take the same observable: the value of the variable z. In fact, already in [5], fractal characteristics of the Lorenz attractor were evaluated with the help of the properly weighted UPO data. Our results are obtained by averaging z for each orbit from the complete set of 111 014 UPOs with $N\leq 20$ turns in the phase space around either of the attractor "wings." For each $N\leq 20$, we also computed 10^6 segments of chaotic trajectories with N turns and calculated values of $\langle z\rangle$ for every such segment.

Comparison of eigenvalues in a set of UPOs with the same N discloses strong inhomogeneity in the distribution of natural measure. Already among 186 orbits with $N=11$ the largest (4618.57) and the smallest (415.59) eigenvalues differ by the factor of 11.1. At $N=19$, there are 27 594 UPOs and the ratio between the extremal eigenvalues is 77.2. Accordingly, the contribution of the "most unstable" UPO is hardly discernible compared to the contribution of the "least unstable" one.

As seen in the left panel of Fig. 1, taking the weights into account shifts and reshapes the distribution of mean values. The solid curve shows the bell-shaped distribution of $\langle z\rangle$ for segments of chaotic orbits with length 20. The dashed curve shows the histogram obtained by summation of $\langle z\rangle$ from all UPOs of the same length. Similarly to Fig. 1 of [1], the maxima of these two curves are shifted with respect to each other. The expectation values of $\langle z\rangle$ for these two distributions are distinctly different: $\overline{\langle z\rangle}$ equals 23.555 for chaotic trajectories and 23.420 for summation over UPOs. This difference, however, almost vanishes for the histogram which incorporates the weights of UPOs: the dotted curve in Fig. 1(a) is much closer to the solid curve and has $\overline{\langle z\rangle}=23.554$. In the case of shorter orbits with length $N=11$ reported in [1], the same effect takes place: ensemble of chaotic orbits predicts $\overline{\langle z\rangle}=23.562$, which is definitely distinct from the value 23.420 obtained by summation over UPOs, but is much closer to the value 23.550 yielded by proper summation with weights.

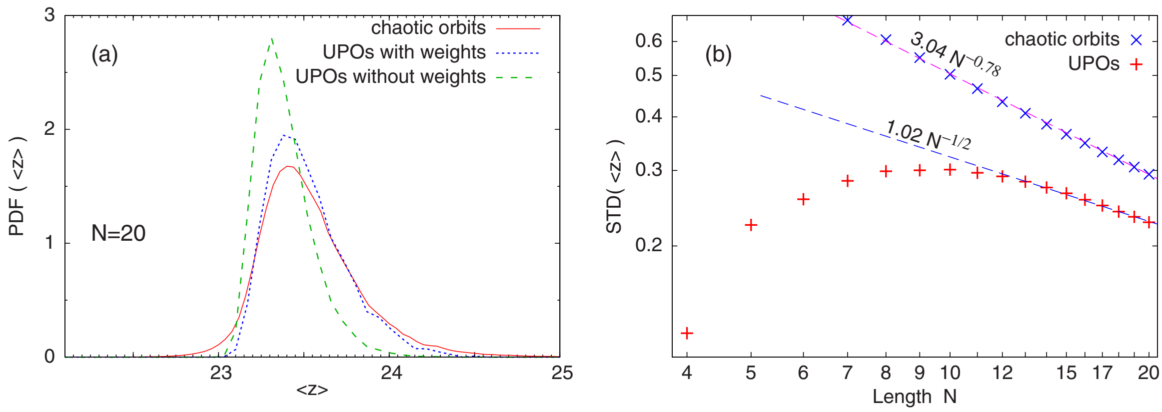

FIG. 1. (Color online) Time averages of $z(t)$ for chaotic trajectories and UPOs of the Lorenz equations. (a) Probability density for orbits of length $N=20$. (Solid line) Average values from 10^6 segments of chaotic trajectories. (Dotted line) Weighted means over all UPOs with $N=20$. (Dashed line) Summation (without weights) over all UPOs with $N=20$. (b) Standard deviation of $\langle z\rangle$ for periodic (pluses) and chaotic (crosses) orbits with N turns. (Dashed lines) Power-law fits.

Obviously, the distribution obtained from finite segments of chaotic orbits cannot be identical to the distribution produced by the set of UPOs of the same length. N rotations in a piece of a chaotic orbit do not necessarily constitute a visit into the neighborhood of an UPO of the same length: they can consist of several visits into vicinities of the shorter UPOs, be a part of the passage near the longer UPO, etc. Therefore—at least in the checked range of N—distributions based on chaotic orbits are broader than their counterparts built from the UPOs data. As shown in the right panel of Fig. 1, the breadth of both distributions decays as a power of $1/N$. For chaotic segments, it approaches the law $3.04N^{-0.78}$ (which nearly coincides with the values from [1]), whereas the standard deviation for a distribution from the set of UPOs with properly assigned weights decays as $1/\sqrt{N}$. Of course, the employed values of N are, at best, moderate and one cannot judge on the asymptotical properties of the dependence. At any rate, however, these data unambiguously show that in the range $N\leq 20$, there are no arguments against eventual convergence to the δ-shaped distribution [7].

Since for small and moderate values of N the distributions are typically broad, it hardly makes sense to discuss whether an attractor of a particular set of equations is accurately modeled by a single UPO of the given length: some of the UPOs are definitely inappropriate, whereas some others may prove to be good. Average values computed along the UPO match the averages along chaotic orbits either in a pathological case when all UPOs except one possess giant eigenvalues (and, hence, negligible statistical weights) or if one deliberately chooses the UPO whose characteristics are close to that of the whole ensemble. For the latter, however, the ensemble (or, at least, its representative parts) should be examined, so there is hardly a gain in the computational efficiency.

The last remark concerns estimates of the topological entropy in [1]. It is known (see, e.g., [6]) that at $r=28$, a chaotic orbit can make not more than 25 consecutive turns around one wing of the Lorenz attractor before a jump to another wing. Once each turn in the half space $x>0$ is coded by "1" and each code in the complementary half space $x<0$ is coded by "0," all binary strings with length $N\leq 25$ are met in the code of a sufficiently long chaotic trajectory. The only missing periodic orbits are those whose symbolic labels consist exclusively of ones or of zeroes. Accordingly, the number of UPOs with the length N for $N\leq 25$ is given by the recursive formula $K(N)=[2^N-2-\Sigma_j jK(j)]/N$, summation being taken along all divisors j of N. However, estimate of the topological entropy as $h_{top}=\lim_{N\to\infty}\sup N^{-1}\log K(N)=\log 2$ is applicable only for the newborn attractor at $r=24.06\ldots$ in which all symbolic strings of arbitrarily large N are encountered. This is not the case for $r=28$: upwards from $N=25$, the tree of symbolic sequences is "pruned" and the growth of number of UPOs as a function of N may become slower. In any case, the range of orbit length $N\leq 14$ employed in [1] for the evaluation of the h_{top} is far too short and hardly appropriate for reliable estimates.

[1] Y. Saiki and M. Yamada, Phys. Rev. E **79**, 015201(R) (2009).
[2] C. Grebogi, E. Ott, and J. A. Yorke, Phys. Rev. A **37**, 1711 (1988).
[3] E. Ott, *Chaos in Dynamical Systems* (Cambridge University Press, 1993).
[4] E. N. Lorenz, J. Atmos. Sci. **20**, 130 (1963).
[5] B. Eckhardt and G. Ott, Z. Phys. B **93**, 259 (1994).
[6] C. Sparrow, *The Lorenz Equations: Bifurcations, Chaos, and Strange Attractors* (Springer, New York, 1982).
[7] In fact, even for the set of unweighted UPOs, the standard deviation of $\langle z\rangle$ decays as $\sim 0.74/\sqrt{N}$. This also implies convergence to the δ-shaped distribution, centered, of course, *not* at the value of the time average.